1 Exercise sheet 01, to be presented on 23.10.2025

1.1 Exc: Atoms, Spins, Photons

In a typical cold atom experiment one deals with atoms with a complex internal structure consisting of many internal states in the presence of external fields such as magnetic fields or laser fields. In order to simplify this system and to identify the relevant states it is useful to consider the hierarchy of relevant energy scales.
Hint: No extensive calculations are required for any of these questions.

(a) A typical laser source for the optical trapping of atoms is a Nd:YAG laser with a wavelength of $\lambda = 1064\,\mathrm{nm}$. What is the photon energy in Joule (J) and electron-Volt (eV), what is the laser frequency?

In many cases one does not need to consider all optical transitions, but can restrict oneself to a single transition connecting the ground state and one excited (electronic) state. This approximation is known as the two-level atom and it will be used often in the lecture. In atomic Ytterbium ($^{174}\mathrm{Yb}$,$m = 173.94u$, $u = 1.66 \times 10^{-27}\,\mathrm{kg}$), the most important optical transition has a wavelength of $\lambda = 398.8\,\mathrm{nm}$. An atom in the excited state will spontaneously decay back into the ground state by emitting a photon, the linewidth of this emission line is approximately $(2\pi)\,29\,\mathrm{MHz}$.

(b) Hypothetical photon-free decay (violates momentum conservation): Suppose that the relaxation would happen without the photon such that the total energy would be converted into kinetic energy. To which temperature would this energy correspond to ($E \simeq k_\mathrm{B}T$)? What would be the final velocity of an atom that was initially at rest?

(c) In reality the atom relaxes via the emission of a photon and it will acquire a velocity due to momentum conservation, the so-called recoil velocity. How big is this velocity and the corresponding kinetic energy (recoil energy)? What transition frequency does this energy correspond to ($\nu = E/h$) and to which temperature would it correspond to? If you keep scattering photons from this atom from the same direction, what can the acceleration maximally be?

The electronic ground state of the atoms is split into sub-states due to the hyperfine interaction and, in the presence of magnetic fields, the Zeeman effect. Most experiments so far have been performed using Alkali atoms (Li, Na, K, Rb, Cs). The common feature of these atoms is the electron configuration, which consists of closed shells (with vanishing angular momentum) and a single outer electron.

(d) In the ground state the outer electron is in an s-orbital with zero orbital angular momentum ($I = 0$). Thus the total angular momentum J of the electron shell is given by the spin of the outer electron $J = S = 1/2$. The nucleus of $^{87}\mathrm{Rb}$ has a nuclear spin of $I = 3/2$. What are the possible angular momenta F for the whole atom? Is this atom a boson or a fermion?

Each of the different hyperfine levels, which are characterized by the total angular momentum F, consists of $2F + 1$ sub-states, characterized by the magnetic quantum number m_F? A typical optical trap will trap all of these equally, but in most cases it is desirable to work with only one or two of these sub-levels.

(e) In order to avoid a redistribution of the atoms among the different sub-states, typically a homogeneous magnetic field on the order of $\mathcal{B} \simeq 1\mathrm{G}$ is applied to lift the degeneracy between the different sub-states. What would be the transition frequency between two neighboring hyperfine Zeeman spin states in a field of $\mathcal{B} = 1\mathrm{G}$, assuming the Rubidium atom is in the $F = 1$ ground state ($g_F = -1/2$)? What would be the corresponding temperature above which thermal energies can flip these spins in atomic collisions?

(f) Why are Alkali atoms so popular in atomic physics?

(g) Draw a sketch of the level scheme (Grotrian diagram) of the ground state $5S_{1/2}$ and excited states $5P$ of $^{87}\mathrm{Rb}$, including the quantum defect, fine-structure, and hyperfine structure. Which numbers for J exist in the $5P$ state? Which numbers for F exist in all the states? Which wavelengths drive the corresponding transitions (i.e. the D_1 and D_2 line)?
Hint: You can use the Alkali Line Data of D. Steck [1].

[1](D. A. Steck. "Rubidium 87 D Line Data". In: *https://steck.us/alkalidata/rubidium87numbers.1.6.pdf* [2008])

(a) Energy

$$E = \hbar\omega = \frac{\hbar 2\pi c}{\lambda} = \quad = \quad 1.77 \times 10^{-19}\,\mathrm{J},$$
$$\frac{E}{1.6 \times 10^{-19}\,\mathrm{C/e}} \quad = \quad 1.1\,\mathrm{eV},$$

and frequency

$$\omega = \frac{2\pi c}{\lambda} = (2\pi) \cdot 282\,\mathrm{THz}.$$

(b) Temperature (with $E = k_B T$)

$$T = \frac{\hbar 2\pi c}{\lambda k_B} = 34\,251\,\mathrm{K}.$$

This temperature is far beyond the temperature of the atoms. This also means that the excited state is only very little occupied in thermal equilibrium with room temperature environment. Also the velocity of the atoms would be very high (with $E = 1/2mv^2$):

$$v = \sqrt{\frac{2\hbar 2\pi c}{m\lambda}} = 1809\,\mathrm{m/s}.$$

(c) For the recoil velocity, we set $\hbar k = mv$ and get:

$$v_r = \frac{\hbar 2\pi}{m\lambda} = 5.5\,\mathrm{mm/s}.$$

The corresponding transition frequency is derived from the so-called recoil energy

$$E_r = \frac{1}{2} m v_r^2 = \frac{\hbar^2 k^2}{2m} = \frac{\hbar^2 (2\pi)^2}{2m\lambda^2} = 4.78 \times 10^{-30}\,\mathrm{J}$$

as:

$$\omega_r = \frac{E_r}{\hbar} = (2\pi) \cdot 721\,\mathrm{kHz}.$$

The corresponding temperature is

$$T = \frac{E_r}{kB} = 0.35\,\mu\mathrm{K}.$$

The maximum acceleration is calculated as $a = \Delta v/\Delta t$, where we set $\Delta v = v_r$, and $\Delta t = \Gamma^{-1}$. The latter is, because there is ~ 1 photon emission per inverse decay rate Γ^{-1} (We neglect here the fact, that the occupation of the excited level in the steady-state is 1/2 max). Thus

$$a = v_r \cdot \Gamma = 1.05 \times 10^6\,\mathrm{m/s^2}.$$

This acceleration equals approximately $10^5 g$!

(d) The total angular momentum $J = 1/2$ of the shell couples to the nuclear spin $I = 3/2$. This leads to possible values of the hyperfine quantum number between $F = |I - J|$ min to $F = |I + J|$ max. This is: $F = 1$ and $F = 2$, i.e. the Rubidium ground state is split into two hyperfine levels. As the quantum number is an integer number, ^{87}Rb is a boson.

(e) The energy of the Zeeman sublevels is given by $\Delta E = g_F m_F \mu_\mathrm{B} \mathcal{B}$, i.e. neighboring Zeeman levels with $\Delta m_F = 1$ in the $F = 1$ ground state with $g_F = 1/2$ are split by $\Delta E = \frac{1}{2}\mu_\mathrm{B}\mathcal{B}$, and the corresponding transition frequency is (consider that 1G=1e-4T)

$$\omega = \frac{\mu_\mathrm{B}\mathcal{B}}{2\hbar} = (2\pi) \cdot 700\,\mathrm{kHz}.$$

That means the transition can be driven with a radio frequency. The corresponding thermal energy would be

$$T = \frac{\mu_\mathrm{B}\mathcal{B}}{2k_\mathrm{B}} = (2\pi) \cdot 34\,\mu\mathrm{K}.$$

That means that atoms should be cooled to temperatures well below that value to avoid spin changing collisions.

(f) That has mainly to do with the fact that alkalis have, as they are hydrogen-like, the simplest level schemes and feature (almost) closed transitions that can be used for laser cooling. Typically, only one additional so-called repump laser is required to pump atoms back to the cooling transition levels. Moreover, the field was intially boosted by the availability of diode laser at 780 nm, i.e. at the D_2 line of Rubidium. This was the case, because all CD players were operated with laser diodes at almost this wavelength.

(g) The nuclear spin is $I = 3/2$. In the ground state $5S_{1/2}$ with $J = 1/2$ we get hyperfine coupling to $F = 1$ and $F = 2$. In the excited $5P$ state, the angular momentum quantum number is $L = 1$. Coupling with the spin of the electron $S = 1/2$ leads to the quantum numbers $J = 1/2$ and $J = 3/2$. Coupling of $5P_{1/2}$ with the nuclear spin leads to $F = 1$ and $F = 2$. Coupling of $5P_{3/2}$ with the nuclear spin leads to $F = 0$, $F = 1$, $F = 2$, and $F = 3$. Due to the quantum defect, the levels $5P$ have a different energy than the level $5S$. The splitting of $5P_{1/2}$ and $5P_{3/2}$ is the fine structure.

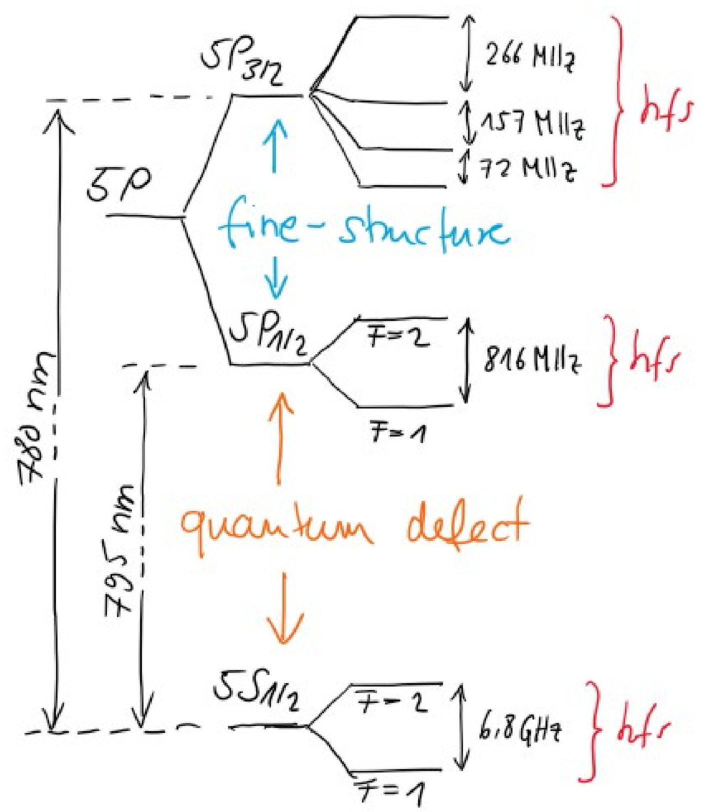

Figure 1: ^{87}**Rb** D1 and D2 line levels

2 Exercise sheet 02, to be presented on 30.10.2025

Solve the following exercises of the script: *Atom-Light Interactions*

Exc.1.3.0.1: Rabi oscillations

Exc.1.3.0.2: Rabi method

Exc.1.3.0.3: Ramsey fringes

Exc.1.3.0.4: Light-shift

Exc.1.3.0.5: Monte Carlo wavefunction simulation of quantum jumps

3 Exercise sheet 03, to be presented on 06.11.2025

Solve the following exercises of the script: *Atom-Light Interactions*

Exc.2.7.0.4: Thermal population of a harmonic oscillator

Exc.2.7.0.12: Photon echo

Exc.2.7.0.15: Rate equations as a limiting case of Bloch equations

Exc.2.7.0.16: Purity of two-level atoms with spontaneous emission

Exc.2.7.0.18: General form of the master equation

4 Exercise sheet 04, to be presented on 13.11.2025

4.1 Exc: Rate equations and scattering cross-section

(a) Use the rate equations

$$\begin{aligned}\frac{dN_1}{dt} &= -N_1\sigma j + N_2\sigma j + \Gamma N_2,\\ \frac{dN_2}{dt} &= N_1\sigma j - N_2\sigma j - \Gamma N_2,\end{aligned}$$

that we derived in the lecture for the two-level system to derive the stationary fraction of atoms in the excited state

$$\frac{N2}{N} = \frac{\sigma j/\Gamma}{1+2\sigma j/\Gamma}.$$

(b) Compare the result for $\frac{N_2}{N}$ above with the result from the optical Bloch equations

$$\frac{N2}{N} = \rho_{22}(\infty) = \frac{s/2}{1+s},$$

from AL(2.78) with saturation parameter

$$s = \frac{2|\Omega|^2}{4\Delta^2+\Gamma^2},$$

from AL(2.77), to derive the frequency-dependent scattering cross-section

$$\sigma(\Delta) = \frac{3\lambda^2}{2\pi}\cdot\frac{\Gamma^2}{4\Delta^2+\Gamma^2}.$$

Hints: Use $j = \frac{I}{\hbar\omega}$, $I = \frac{1}{2}\cdot\varepsilon_0 c E_0^2$, $\Omega = \frac{d\cdot E_0}{\hbar}$, and $\Gamma = \frac{\omega^3 d^2}{3\pi\varepsilon_0\hbar c^3}$, AL(2.34).

Solution:

(a) We set $\frac{dN_2}{dt} = 0$:

$$N_1\sigma j - N_2\sigma j - \Gamma N_2 = 0,$$

from which we get

$$N_1\sigma j = (\Gamma+\sigma j)N_2.$$

Now, we set $N_1 = N - N_2$ and solve for N_2:

$$N_2(\Gamma + 2\sigma j) = N\sigma j,$$

from which we get

$$\frac{N_2}{N} = \frac{\sigma j}{\Gamma+2\sigma j} = \frac{\sigma j/\Gamma}{1+2\sigma j/\Gamma}.$$

(b) From

$$\frac{\sigma j/\Gamma}{1+2\sigma j/\Gamma} = \frac{s/2}{1+s},$$

we derive that

$$\sigma j/\Gamma = s/2 = \frac{|\Omega|^2}{4\Delta^2+\Gamma^2},$$

where we put in the definition of s in the second step. Respectively

$$\sigma = \frac{|\Omega|^2}{4\Delta^2+\Gamma^2}\cdot\frac{\Gamma}{j}.$$

We now use that $j = I/(\hbar\omega)$, and

$$I = \frac{1}{2} \cdot \varepsilon_0 c E_0^2 = \frac{1}{2} \cdot \varepsilon_0 c \cdot \frac{\hbar^2 |\Omega|^2}{d^2},$$

where we used the definition of the Rabi frequency $\Omega = \frac{d \cdot E_0}{\hbar}$ in the last step. Now, we replace the dipole matrix element by $d^2 = 3\pi\varepsilon_0 \hbar c^3 \Gamma/\omega^3$ (as derived from AL(2.34)), and get

$$I = \frac{1}{2} \cdot \varepsilon_0 c \cdot \frac{\hbar^2 |\Omega|^2 \omega^3}{3\pi\varepsilon_0 \hbar c^3 \Gamma} = \frac{2\pi\hbar\omega|\Omega|^2}{3\Gamma\lambda^2},$$

and

$$j = \frac{I}{\hbar\omega} = \frac{2\pi|\Omega|^2}{3\Gamma\lambda^2},$$

which we plug in the equation for σ:

$$\sigma = \frac{|\Omega|^2}{4\Delta^2 + \Gamma^2} \cdot \frac{\Gamma \cdot 3\Gamma\lambda^2}{2\pi|\Omega|^2} = \frac{3\lambda^2}{2\pi} \cdot \frac{\Gamma^2}{4\Delta^2 + \Gamma^2}.$$

4.2 Exc: Broadening effects

(a) Compare the natural linewidth of the ^{87}Rubidium D2 line with the Doppler and pressure broadening at room temperature ($T = 20\,°\mathrm{C}$, Rb mass $1 \times 10^{-25}\,\mathrm{kg}$). You can use the Steck alkali data sheet (Figure 1) to get the vapor pressure and thus the Rubidium density, and use the self-pressure broadening per density, $\beta = (2\pi) \cdot 7 \times 10^{-8}\,\mathrm{Hz \cdot cm^3}$, to calculate the value of pressure broadening.

(b) What are the numbers for Doppler and pressure broadening in a cold atom cloud with typical values after laser cooling of $T = 10\,\mu\mathrm{K}$ and $n = 1 \times 10^{12}\,\mathrm{cm^{-3}}$?

(c) At which laser intensity is the saturation broadening (i) twice as large as the natural linewidth, and (ii) as large as the Doppler broadening of the thermal room temperature gas (calculated above)?

(d) To which diameter can a Gaussian laser beam be focused in a thermal vapour cell at room temperature such that the transit time broadening does not exceed the natural linewidth?

Solution:

(a) The natural linewidth is $\Gamma = (2\pi) \cdot 6\,\mathrm{MHz}$.
The Doppler width is given by

$$\Gamma_D = 2k\sqrt{\frac{2k_BT}{m}} = \frac{4\pi}{780\,\mathrm{nm}}\sqrt{\frac{2 \cdot 1.38 \times 10^{-23}\,\mathrm{J/K} \cdot 293\,\mathrm{K}}{1 \times 10^{-25}\,\mathrm{kg}}} = (2\pi) \cdot 729\,\mathrm{MHz}.$$

From Steck, we get a vapour pressure of $p = 2 \times 10^{-7}\,\mathrm{mbar}$, and from that using the ideal gas law a density of

$$n = \frac{p}{k_BT} = \frac{2 \times 10^{-5}\,\mathrm{Pa}}{1.38 \times 10^{-23}\,\mathrm{J/K} \cdot 293\,\mathrm{K}} = 5 \times 10^{15}\,\mathrm{m^{-3}} = 5 \times 10^{9}\,\mathrm{cm^{-3}},$$

from which the self-pressure broadening follows as

$$\Gamma_p = \beta n = (2\pi) \cdot 7 \times 10^{-8}\,\mathrm{Hz \cdot cm^3} \cdot 5 \times 10^{9}\,\mathrm{cm^{-3}} = (2\pi) \cdot 350\,\mathrm{Hz}.$$

The pressure broadening is thus negligible. Note that pressure broadening may become important, if a buffer gas (noble gases are often used) is mixed with the Rb vapour. This is often done in order to limit the motion of the Rb atoms by scattering with the buffer gas atoms.

(b) For the laser-cooled atoms, we get

$$\Gamma_D = 2k\sqrt{\frac{2k_BT}{m}} = \frac{4\pi}{780\,\mathrm{nm}}\sqrt{\frac{2 \cdot 1.38 \times 10^{-23}\,\mathrm{J/K} \cdot 10 \times 10^{-6}\,\mathrm{K}}{1 \times 10^{-25}\,\mathrm{kg}}} = (2\pi) \cdot 135\,\mathrm{kHz}.$$

The line is thus no longer Doppler broadened. The self-pressure broadening is now with

$$\Gamma_p = \beta n = (2\pi) \cdot 7 \times 10^{-8}\,\mathrm{Hz \cdot cm^3} \cdot 1 \times 10^{12}\,\mathrm{cm^{-3}} = (2\pi) \cdot 700\,\mathrm{kHz}$$

comparable with the Doppler broadening, but also much smaller than the natural linewidth.

(c) We use the formula for the saturation broadening

$$\delta_{\text{fwhm}} = \Gamma\sqrt{1 + \frac{I}{I_{\text{sat}}}}$$

and solve for

$$I = \left[\left(\frac{\delta_{\text{fwhm}}}{\Gamma}\right)^2 - 1\right] \cdot I_{\text{sat}} = \left[\left(\frac{729\,\text{MHz}}{6\,\text{MHz}}\right)^2 - 1\right] \cdot 1.6\,\text{mW/cm}^2 = 23\,618\,\text{mW/cm}^2,$$

where the value of the Doppler width was taken from above. If the Doppler-broadened linewidth is given by $\delta_{\text{fwhm}} = 2\Gamma$, we get

$$I = 3 \cdot I_{\text{sat}} = 4.8\,\text{mW/cm}^2.$$

(d) We use the formula from the lecture for the transit-time broadening and solve for the Gaussian beam diameter: $d = 4.7\frac{\bar{v}}{\delta_{\text{fwhm}}}$. We use for the thermal velocity

$$\bar{v} = \sqrt{\frac{3k_BT}{m}} = \sqrt{\frac{3 \cdot 1.38 \times 10^{-23}\,\text{J/K} \cdot 293\,\text{K}}{1 \times 10^{-25}\,\text{kg}}} = 348\,\text{m/s},$$

and set $\delta_{\text{fwhm}} = \Gamma$ to get

$$d = 4.7 \cdot \frac{348\,\text{m/s}}{(2\pi) \cdot 6\,\text{MHz}} = 44\,\mu\text{m}.$$

4.3 Exc: Transmission through vapour cell

We want to calculate the transmission of a light beam through a typical vapour cell of length $l = 10\,\text{cm}$ with Rubidium ($\lambda = 780\,\text{nm}$, $\Gamma = (2\pi) \cdot 6\,\text{MHz}$, $T = 293\,\text{K}$), and assume that the probe beam has little intensity, i.e. saturation is negligible.

(a) Which fraction of the incoming light is transmitted through the cell on resonance? Use the Lambert-Beer law and assume that all atoms are participating. Calculate the atomic density with the ideal gas law.

(b) The value that we just calculated is way too small. In experiment, some ten percent of the incoming light are transmitted through the cell. The problem with the calculation above is, that we assumed that all atoms are contributing to the absorption. This is not the case in a Doppler broadened spectrum. Instead, for given laser wavelength (we neglect here the bandwidth of the laser), only atoms with some velocities within a range Δv_z are resonant. Calculate Δv_z by setting the Doppler width of that range equal to the natural linewidth Γ. In the next step, calculate which fraction of all atoms (N/N_{tot}) is interacting with the laser. Use the atomic velocity distribution,

$$N(v_z)dv_z = N_{\text{tot}} \cdot \frac{1}{\sqrt{\pi}v_0} \cdot \exp(-\frac{v_z^2}{v_0})dv_z,$$

with thermal velocity $v_0 = \sqrt{\frac{2k_BT}{m}}$. Assume that the laser frequency is on resonance with the transition (i.e. $v_z = 0$), and solve the integral in first order:

$$N = \int_0^{\Delta v_z} N(v_z)dv_z \approx N(0) \cdot \Delta v_z.$$

Now, use a atomic density which is reduced by this factor to recalculate the transmission through the vapour cell with the Lambert-Beer law.

(c) Now we do saturation spectroscopy and want to estimate the size of the Lamb dip. The transmission of the probe beam alone through the vapour cell is T (this is the bottom of the Doppler-broadened absorption spectrum). By how much does the transmission change when the saturation beam in included? Calculate the corresponding ΔT as a function of T. Assume that the saturation beam completely saturates the transition, and the probe beam has an intensity far below saturation such that its contribution to the excitation of the excited state is negligible. For which transmission T reaches the Lamb dip its maximum size, and how big is it?

Solution:

(a) From the vapour pressure of $p = 2 \times 10^{-7}$ mbar (Steck) we calculate the atomic density of

$$n = \frac{p}{k_B T} = \frac{2 \times 10^{-5}\,\mathrm{Pa}}{1.38 \times 10^{-23}\,\mathrm{J/K} \cdot 293\,\mathrm{K}} = 5 \times 10^{15}\,\mathrm{m}^{-3} = 5 \times 10^{9}\,\mathrm{cm}^{-3},$$

using the ideal gas law. The resonant scattering cross section is given by $\sigma = \frac{3}{2\pi}\lambda^2 = 2.9 \times 10^{-13}\,\mathrm{m}^2$. Thus, the transmission is with the Lambert-Beer law given by

$$\frac{I}{I_0} = \exp(-n \cdot \sigma \cdot l) = \exp(-5 \times 10^{15}\,\mathrm{m}^{-3} \cdot 2.9 \times 10^{-13}\,\mathrm{m}^2 \cdot 0.1\,\mathrm{m}) = 1 \times 10^{-63}.$$

Thus, there is no light coming through the vapour cell, in contrast to what the experiment shows us.

(b) We set $\Delta_D = k \cdot \Delta v_z = \Gamma$ and solve for the velocity range:

$$\Delta v_z = \frac{\Gamma}{k} = \frac{\Gamma \cdot \lambda}{2\pi} = 6\,\mathrm{MHz} \cdot 780 \times 10^{-9}\,\mathrm{m} = 4.68\,\mathrm{m/s}.$$

Thus, a laser with fixed frequency will excite atoms within this velocity range. The integral gives with $v_z = 0$:

$$\frac{N}{N_{\mathrm{tot}}} = \frac{\Delta v_z}{\sqrt{\pi} v_0}.$$

We calculate the thermal velocity as

$$v_0 = \sqrt{\frac{2 k_B T}{m}} = \sqrt{\frac{2 \cdot 1.38 \times 10^{-23}\,\mathrm{J/K} \cdot 293\,\mathrm{K}}{1.44 \times 10^{-25}\,\mathrm{kg}}} = 237\,\mathrm{m/s}.$$

This gives us a fraction of all atoms of

$$\frac{N}{N_{\mathrm{tot}}} = \frac{4.68\,\mathrm{m/s}}{\sqrt{\pi} \cdot 237\,\mathrm{m/s}} = 0.011.$$

Thus, only one percent of all atoms are interacting with the laser field. We can attribute this reduction also the the density of atoms in the Lambert-Beer law and calculate the transmission:

$$\frac{I}{I_0} = \exp(-0.011 \cdot n \cdot \sigma \cdot l) = \exp(-0.011 \cdot 5 \times 10^{15}\,\mathrm{m}^{-3} \cdot 2.9 \times 10^{-13}\,\mathrm{m}^2 \cdot 0.1\,\mathrm{m}) = 0.20.$$

Thus, the intensity after the cell is reduced to twenty percent. This is very good for spectrocopy. Note that with multiple transitions, and considering the corresponding Clebsch-Gordan coefficients, the transmission will be typically higher by several tens of percent.

(c) For the probe beam alone, the transition is

$$T = \exp(-n\sigma l) = \exp(-x),$$

where we have introduced $x = n\sigma l$. The atom density n is the density of atoms in the ground state, which, here equals the overall density of atoms (in the respective velocity class), as the weak probe does not excite atoms to the excited state. The strong saturation beam however excites half of the atoms to the excited state. By that, half of the atoms are missing in the ground state. The new transmission is

$$T' = \exp(-\frac{n}{2}\sigma l) = \exp(-\frac{x}{2}),$$

and thus:

$$\Delta T = T' - T = \exp(-\frac{x}{2}) - \exp(-x).$$

We calculate the maximum ΔT by differentiation

$$\frac{d\Delta T}{dx} = -\frac{1}{2}\exp(-\frac{x}{2}) + \exp(-x) = 0,$$

which results in

$$x = 2\ln(2).$$

The corresponding transmission is

$$T = \exp(-x) = \exp(-2\ln(2)) = 25\%,$$

and the change in transmission is

$$\Delta T = \exp(-x/2) - \exp(-x) = \exp(-\ln(2)) - \exp(-2\ln(2)) = 25\%.$$

That means that on the peak of the Lamb dip the transmission increases from 25% to 50%.

4.4 Exc: Spectroscopy of a Λ-system

We have learned in the lecture that saturation spectroscopy with several excited state leads to so-called cross-over resonances, where an additional Lamb dip like feature appears at the center frequencies between the resonances. These dips are more pronounced than the Lamb-dips at the resonance frequencies. We want to investigate here the opposite situation of several ground states. For simplicity, we consider two ground states at different frequencies with a separation smaller than the Doppler width of the thermal gas. Like in saturation spectroscopy, a strong saturation beam is traversing the vapour cell, opposite to a weak probe beam. How does the spectrum look like in this situation? Make a sketch! Argue qualitatively without calculations. Hint: In thermal equilibrium (without pumping), the two ground states are equally populated. How does optical pumping change these populations?**Solution:** In principle, the same arguments hold as for standard saturation spectroscopy: If the frequency of the saturation and probe laser is resonant with one of the two atomic transition frequencies, both lasers address atoms with velocity $v_z = 0$, i.e., there is saturation, which means that less atoms are in the ground state, and thus, the probe laser is absorbed less. This gives rise to a Doppler-free Lamb dip at each transition frequency. Now, we tune the laser frequency to the center between the two transition frequencies. For atoms with velocity class $v_z = \omega_{12}/k$, where ω_{12} is the frequency difference between the two ground states, the saturation beam is resonant from the first ground state to the excited state, and the probe laser is resonant from the second ground state to the excited state. The saturation beam does in this situation, however, not deplete the probed, second ground state; it excites atoms to the excited state from where they can decay back into both ground states. Atoms that decay back to the first ground state are then reexcited by the saturation beam. Atoms that decay to the second ground state, could be excited only by the weak probe beam. Thus, they will stay in the second ground state. Effectively, atoms are pumped from the first to the second ground state. The probe beam interacts now with more atoms than without the saturation beam, and can be absorbed more. The sign of the cross-over resonance is reverted, compared to normal saturation spectroscopy. (The same is true for the begative velocity class, but with opposite roles.)

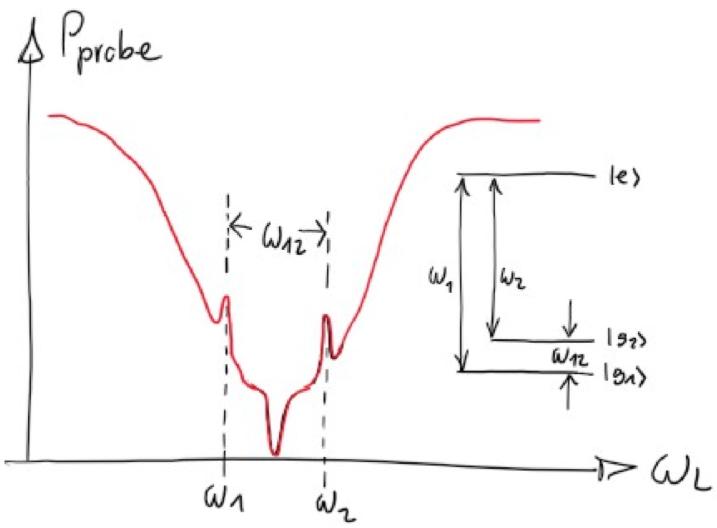

Figure 2: Saturation spectroscopy with two ground states

5 Exercise sheet 05, to be presented on 20.11.2025

5.1 Exc: Laser design

Let's assume, we have a laser medium with quantum defect $Q = \frac{\omega}{\omega_P} = 0.8$, pump efficiency $\eta = 0.5$, and saturation power of $P_{\text{sat}} = 5\,\text{mW}$. The laser resonator has round-trip losses of $L = 5\%$. The pump source has a maximum power of $P_{\text{pump}}^{\text{max}} = 100\,\text{mW}$. We want to determine numerically the output coupler transmission T, for which the laser reaches a maximum output power at maximum pumping. For that, write a program that plots the output power P_{out} vs. T. Use the optimum value of T determined from the graph to plot the output power P_{out} vs. the pump power P_{pump}. Plot in the same graph the corresponding curves for transmissions T that are 20% bigger and 20% smaller, respectively.
Solution: We use following equation for the output power:

$$P_{\text{out}} = \varepsilon(P_{\text{pump}} - P_{\text{th}}),$$

with slope efficiency

$$\varepsilon = \eta Q \cdot \frac{T}{1 - Rm},$$

and threshold power

$$P_{\text{th}} = \frac{1}{\eta Q} \cdot \frac{1 - R_m}{Rm} \cdot P_{\text{sat}},$$

where the round-trip reflectivity is given by

$$R_m = (1 - T) \cdot (1 - L).$$

In the program, the transmission goes from zero to one in steps of 0.01. The result is shown in Fig. 3. From the left subfigure we deduce an optimum transmission of $T = 36\%$ with an output power of 34 mW.

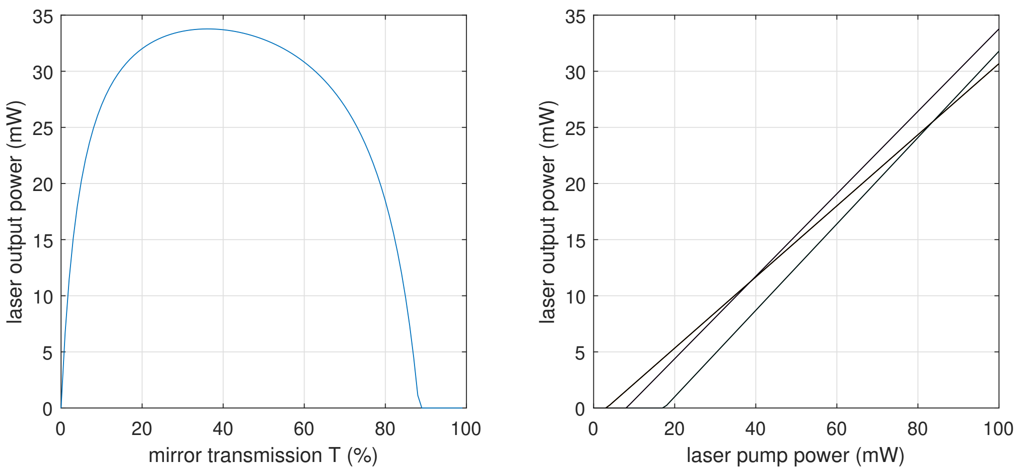

Figure 3: Simulated laser output power

Smaller and larger transmission lead to less output power, as shown in the right subfigure.

5.2 Exc: Ray optics

(a) Use the ABCD-matrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{n_1 - n_2}{Rn_2} & \frac{n_1}{n_2} \end{pmatrix}$$

for an interface from a medium with refractive index n_1 to a medium with n_2 and curvature $R = \infty$ (flat surface) of the interface to derive the Snellius law $n_1 \sin(\alpha_1) = n_2 \sin(\alpha_2)$

(b) Calculate the effective ABCD matrix for transmission through a glass plate (thickness d and refractive index n) with plane-parallel surfaces.

(c) What is the effective focal length of an interface from (i) air to glass and (ii) glass to air with curvature R? Use $n = 1.0$ and $n = 1.5$, respectively.

Solution:

(a) For a flat surface, the ABCD matrix reads

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{n_1}{n_2} \end{pmatrix}$$

We apply the matrix to a ray vector with position $x_1 = 0$ and slope x'_1 and get

$$\begin{pmatrix} x_2 \\ x'_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{n_1}{n_2} \end{pmatrix} \begin{pmatrix} 0 \\ x'_1 \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{n_1}{n_2} x'_1 \end{pmatrix}.$$

We also note that the slope $x'_j = \tan(\alpha_j) \approx \sin(\alpha_j)$ for small angles (paraxial approximation), for

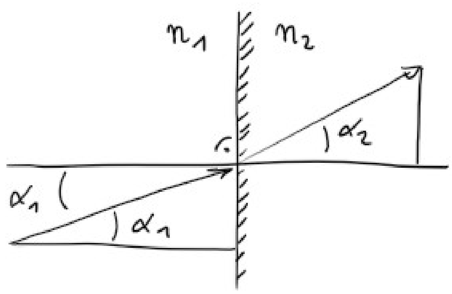

Figure 4: Refraction at an interface

which the ABCD matrix is valid.. Thus, the equation $x'_2 = \frac{n_1}{n_2} x'_1$ is equivalent to the Snellius law $n_1 \sin(\alpha_1) = n_2 \sin(\alpha_2)$.

(b) We now calculate the transmission through the glass plate by multiplication of the ABCD matrices M_1: interface from air ($n_1 = 1$) to glass ($n_2 = 1.5$), M_2: free propagation by distance d, and M_3: interface from glass to air:

$$\begin{aligned} M = M_3 \cdot M_2 \cdot M_1 &= \begin{pmatrix} 1 & 0 \\ 0 & \frac{n_2}{n_1} \end{pmatrix} \cdot \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & \frac{n_1}{n_2} \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & \frac{n_2}{n_1} \end{pmatrix} \begin{pmatrix} 1 & d\frac{n_1}{n_2} \\ 0 & \frac{n_1}{n_2} \end{pmatrix} \\ &= \begin{pmatrix} 1 & d\frac{n_1}{n_2} \\ 0 & 1 \end{pmatrix}, \end{aligned}$$

which has the form of a free propagation with effective distance $d\frac{n_1}{n_2}$. In particular, when $n_1 = 1$, as for air, the effective length of the glass plate is reduced to d/n_2. Thus, the laser beam line is compressed in the glass, which is physically caused by the refraction at the surfaces.

(c) To compare the single curved interface with the action of a lens with focal length f, we assume that a beam is incident on the surface at distance d to the optical axis, but parallel to it. We start with the case (i), where $n_1 = 1$, and $n_2 = n = 1.5$:

$$\vec{x}_2 = \begin{pmatrix} 1 & 0 \\ \frac{n_1 - n_2}{n_2 R} & \frac{1}{n_2} \end{pmatrix} \begin{pmatrix} d \\ 0 \end{pmatrix} = \begin{pmatrix} d \\ \frac{n_1 - n_2}{n_2 R} d \end{pmatrix} = \begin{pmatrix} d \\ \frac{-0.5}{1.5R} d \end{pmatrix} = \begin{pmatrix} d \\ -\frac{1}{3R} d \end{pmatrix}$$

We compare this to the action of a lens with focal lenth f:

$$\vec{x}_2 = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} \begin{pmatrix} d \\ 0 \end{pmatrix} = \begin{pmatrix} d \\ -\frac{1}{f}d \end{pmatrix},$$

from which we immediately see that the focal length of the interface is $f = 3R$. For positive radius R, the action of the interface is that of a focussing lens, for negative radius, it's that of a diverging lens. In case (ii), the roles of n_1 and n_2 are interchanged, which turn the sign of the entry C of the ABCD matrix. Thus, a positive radius now corresponds to a diverging lens, and a negative radius to a focussing lens.

Extension of this exercise: If one adds a second curved interface, one can simulate a biconvex thin lens. Here, the first interface has positive radius R, and the second interface negative radius $-R$. One has to take into account that on the first interface $n_1 = 1$ and $n_2 = n$, whereas at the second interface $n_1 = n$, and $n_2 = 1$. We multiply the ABCD matrices:

$$\begin{pmatrix} 1 & 0 \\ \frac{n-1}{-R} & n \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ \frac{1-n}{nR} & \frac{1}{n} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{2(1-n)}{R} & 1 \end{pmatrix}.$$

Thus, one finds that $f = \frac{R}{2(n-1)} = R$ for $n = 1.5$.

5.3 Exc: Paraxial Helmholtz equation and its fundamental solution

(a) Derive the paraxial Helmholtz equation by putting the ansatz

$$u(\vec{r}) = \Psi(x, y, z) \cdot \exp(-ikz)$$

into the Helmholtz equation

$$\vec{\nabla}^2 u(\vec{r}) + k^2 u(\vec{r}) = 0,$$

and neglect terms $\propto \frac{\partial^2}{\partial z^2}$.

(b) Solve the paraxial Helmholtz equation

$$\left[2ik\frac{\partial}{\partial z} - \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\right]\Psi = 0, \tag{1}$$

to derive the fundamental Gauss mode: First, plug the ansatz

$$\Psi = e^{-i\left[\varphi(z) + k(x^2+y^2)/(2q(z))\right]} \tag{2}$$

into (1) to derive an equation for q, $q' = \frac{\partial q}{\partial z}$, φ, and $\varphi' = \frac{\partial \varphi}{\partial z}$. You will find the conditions

$$\begin{aligned} q' &= 1, \\ \varphi' &= \frac{-i}{q}. \end{aligned}$$

Then, integrate q' to get a (rather simple) expression for $q(z)$. You can set $q(z = 0) = q_0$, with q_0 the q-parameter at the focus of the laser beam. As next step, use the definition

$$\frac{1}{q(z)} \equiv \frac{1}{R(z)} - i\frac{\lambda}{\pi w(z)^2}, \tag{3}$$

to make the ansatz (2) equal the usual definition of the Gaussian mode with

$$\Psi = e^{-i\varphi - ik\frac{x^2+y^2}{2R^2} - \frac{x^2+y^2}{w^2}},$$

where the second term in the exponent describes the curvature of the wave fronts, and the third term the Gaussian shape in the transverse direction. Now, use (3) and the expression that you got for $q(z)$ to derive an equation that relates $R(z)$, $w(z)$ and $w_0 \equiv w(0)$. In this equation, $R(0)$

does not appear, as the wave fronts in the focus are plane, such that you can set $R(0) = \infty$. As a consequence, it drops out of the equation. Separate the equation for real and imaginary parts, which have to be fulfilled, separately, to derive equations for $w(z)$ and $R(z)$. Finally, integrate the equation for φ' to get $\varphi(z)$. For solving the integral, you can resort to following relations

$$\begin{aligned} \int \frac{z}{a^2+z^2} dz &= \frac{1}{2}\ln(a^2+z^2) + C, \\ \int \frac{1}{a^2+z^2} dz &= \frac{1}{a}\arctan\left(\frac{z}{a}\right) + C. \end{aligned}$$

Now, plug everything into the ansatz (2) to show that the result equals the Gaussian fundamental mode.

Solution:

(a) Calculate

$$\begin{aligned} \vec{\nabla}^2 u(\vec{r}) &= \left(\partial_x^2 + \partial_y^2 + \partial_z^2\right)\left[\Psi(x,y,z)\cdot\exp(-ikz)\right] \\ &= \left[\partial_x^2\Psi + \partial_y^2\Psi + \partial_z^2\Psi\right]\cdot\exp(-ikz) + \Psi\cdot(-ik)^2\exp(-ikz) + 2\cdot(\partial_z\Psi)(-ik)\exp(-ikz). \end{aligned}$$

We neglect the term $\propto \partial_z^2\Psi$, as Ψ is slowly varying in the z-direction. Moreover, the term with the second derivative of the exponential function, i.e. the term $\Psi\cdot(-ik)^2\exp(-ikz) = -k^2\Psi\exp(-ikz) = -k^2 u$ cancels with the second term of the Helmholtz equation. Finally, one divides the Helmholtz equation by $\exp(-ikz)$ and arrives at the paraxial Helmholtz equation

$$\left[2ik\frac{\partial}{\partial z} - \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\right]\Psi(x,y,z) = 0.$$

(b) Inserting the ansatz

$$\psi = e^{-\imath\left[\varphi(z) + k(x^2+y^2)/2q(z)\right]} \tag{4}$$

into the paraxial Helmholtz equation, we obtain,

$$\begin{aligned} 0 = 2\imath k e^{-\imath[\varphi + k(x^2+y^2)/2q]}\left(-\imath\frac{\partial\varphi}{\partial z} + \frac{\imath k(x^2+y^2)}{2q^2}\frac{\partial q}{\partial z}\right) - 2e^{-\imath[\varphi+k(x^2+y^2)/2q]}\frac{-\imath k}{q} \\ - e^{-\imath[\varphi+k(x^2+y^2)/2q]}\left(\frac{-\imath kx}{q}\right)^2 - e^{-\imath[\varphi+k(x^2+y^2)/2q]}\left(\frac{-\imath ky}{q}\right)^2 . \end{aligned} \tag{5}$$

This leads directly to the equation,

$$0 = (q'-1)\frac{\imath k(x^2+y^2)}{q^2} - 2\imath\varphi' + \frac{2}{q} . \tag{6}$$

For Eqn. (6) to be valid at all x and y, we need $q' = 1$ and $\varphi' = \frac{-\imath}{q}$. Integrating q', we find,

$$q(z) = q_0 + z . \tag{7}$$

It is practical to introduce real beam parameters

$$\boxed{\frac{1}{q} \equiv \frac{1}{R} - \imath\frac{\lambda}{\pi w^2}} . \tag{8}$$

Inserting this into the ansatz (4),

$$\psi = e^{-\imath\varphi - \imath\frac{k(x^2+y^2)}{2R^2} - \frac{(x^2+y^2)}{w^2}} , \tag{9}$$

it becomes clear that $R(z)$ is the radius of curvature and $w(z)$ is the diameter of the beam. Evaluating q_0 at the position of the focus (beam waist), where $R = \infty$, we get from (7) along with the definition (8),

$$\frac{1}{\frac{1}{R_z} - \imath\frac{\lambda}{\pi w_z^2}} = q(z) = q_0 + z = \frac{1}{\frac{1}{\infty} - \imath\frac{\lambda}{\pi w_0^2}} + z = \imath\frac{\pi w_0^2}{\lambda} + z , \tag{10}$$

The separation of this result into a real part and an imaginary part gives,

$$\frac{z}{R}+\frac{w_0^2}{w^2}=1 \quad \text{and} \quad \frac{\pi w^2}{\lambda R}=\frac{\lambda z}{\pi w_0^2} \ . \tag{11}$$

Solving the second equation for $1/R$ and replacing this in the first equation gives an equation for w,

$$\boxed{w^2=w_0^2\left[1+\left(\frac{\lambda z}{\pi w_0^2}\right)^2\right]} \ . \tag{12}$$

This expression can now be replaced in the second equation,

$$\boxed{R=z\left[1+\left(\frac{\pi w_0^2}{\lambda z}\right)^2\right]} \ . \tag{13}$$

We call $z_R \equiv q_0$ the Rayleigh length. Now we integrate φ',

$$\begin{aligned}\varphi &= \int_0^z \frac{-\imath}{q}dz=\int_0^z \frac{-\imath dz}{\imath z_R+z}=-\imath\int_0^z \frac{zdz}{z_R^2+z^2}-\int_0^z \frac{z_R dz}{z_R^2+z^2} \\ &= -\frac{\imath}{2}\ln\frac{z_R^2+z^2}{z_R^2}-\arctan\frac{z}{z_R}=-\imath\ln\frac{w}{w_0}-\arctan\frac{\lambda z}{\pi w_0^2} \ .\end{aligned} \tag{14}$$

Hence,

$$\boxed{\psi(\mathbf{r})=\frac{w_0}{w}e^{\imath\arctan(-z/q_0)-\imath k(x^2+y^2)/2q}} \ . \tag{15}$$

5.4 Exc: Gaussian laser beams

(a) Imagine you want to send a laser beam with a wavelength of 532 nm to small retro-reflectors (cat-eyes) on the moon in order to measure its distance. How large would the beam waist of this laser have to be such that it has expanded on its way to the moon only by a factor of $\sqrt{2}$? What fraction of the light power can be realistically expected to be reflected from the moon, when the emitted beam has a beam waist of $w_0 = 5\,\text{cm}$? The overall area of the retro-reflectors positioned on the moon is $A_d = 0.25\,\text{m}^2$.

(b) Prove, that with the definition of how the change of the q-parameter is calculated from the ABCD-matrix,

$$q_2=\frac{A_1q_1+B_1}{C_1q_1+D_1},$$

the total action of two ABCD matrices can be calculated by matrix multiplication, i.e., if

$$q_3=\frac{A_2q_2+B_2}{C_2q_2+D_2},$$

then

$$q_3=\frac{Aq_1+B}{Cq_1+D},$$

with

$$\begin{pmatrix}A & B\\ C & D\end{pmatrix}=\begin{pmatrix}A_2 & B_2\\ C_2 & D_2\end{pmatrix}\cdot\begin{pmatrix}A_1 & B_1\\ C_1 & D_1\end{pmatrix}$$

(c) Let a collimated beam with $R = \infty$ and beam waist w be focussed by a thin lens with focal length f. Calculate with the help of the ABCD matrix method, at which distance after the lens a focus is formed, and calculate its beam waist. The ABCD matrices of the lens, and of free propagation are

$$M_f=\begin{pmatrix}1 & 0\\ -1/f & 1\end{pmatrix}; \qquad M_x=\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix},$$

respectively. Derive simpler formulas which are valid in the limit of $w \gg \lambda$.

Solution:

(a) For an expansion by $\sqrt{2}$, the Rayleigh length $z_0 = \frac{\pi w_0^2}{\lambda}$ should equal the distance d to the moon. Thus,

$$w_0 = \sqrt{\frac{d \cdot \lambda}{\pi}} = \sqrt{\frac{3 \times 10^8\,\mathrm{m} \cdot 532\,\mathrm{nm}}{\pi}} = 7\,\mathrm{m}.$$

Now, we calculate the beam waist on the moon after expansion with $w_0 = 5\,\mathrm{cm}$:

$$w(d) = w_0\sqrt{1 + \left(\frac{d}{z_0}\right)^2} = w_0\sqrt{1 + \left(\frac{d \cdot \lambda}{\pi w_0^2}\right)^2} \approx 1\,\mathrm{km}.$$

For simplicity, we calculate the ratio

$$\frac{A_d}{\pi w(d)^2} = \frac{0.25\,\mathrm{m}^2}{\pi(1 \times 10^3\,\mathrm{m})^2} \approx 1 \times 10^{-7}\,,$$

without taking the Gaussian profile into account. Consider, that only a very small fraction of the reflected light can be finally detected back on earth.

(b) First calculate the matrix

$$\begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix} \cdot \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix} = \begin{pmatrix} A_1A_2 + C_1B_2 & B_1A_2 + D_1B_2 \\ A_1C_2 + C_1B_2 & B_1C_2 + D_1D_2 \end{pmatrix}.$$

Now put

$$q_1 = \frac{A_1q_0 + B_1}{C_1q_0 + D_1}$$

into

$$q_2 = \frac{A_2q_1 + B_2}{C_2q_1 + D_2} = \frac{A_2\frac{A_1q_0+B_1}{C_1q_0+D_1} + B_2}{C_2\frac{A_1q_0+B_1}{C_1q_0+D_1} + D_2} = \frac{A_2(A_1q_0 + B_1) + B_2(C_1q_0 + D_1)}{C_2(A_1q_0 + B_1) + D_2(C_1q_0 + D_1)}$$
$$= \frac{(A_1A_2 + C_1B_2)q_0 + (B_1A_2 + D_1B_2)}{(A_1C_2 + C_1D_1)q_0 + (B_1C_2 + D_1D_2)}.$$

We see the the parts in brackets equal the corresponding entries of the full matrix.

(c) We assume that the beam waist before the lens is w, and that the beam is collimated, i.e. $R = \infty$. Thus the initial q-parameter is

$$\frac{1}{q_1} = \frac{1}{R} - i\frac{\lambda}{\pi w^2} = -i\frac{\lambda}{\pi w^2}, \tag{16}$$

which leads to

$$q_1 = \frac{i\pi w^2}{\lambda}. \tag{17}$$

After the lens with focal length f the q-parameter is given by

$$q_2 = \frac{q_1}{-\frac{q_1}{f} + 1} = \frac{\frac{i\pi w^2}{\lambda}}{-\frac{i\pi w^2}{\lambda f} + 1}. \tag{18}$$

And after a propagation in free space with length l, it is

$$q_3 = q_2 + L = \frac{\frac{i\pi w^2}{\lambda}\left(1 - \frac{L}{f}\right) + L}{-\frac{i\pi w^2}{\lambda f} + 1}. \tag{19}$$

We calculate the inverse and separate real and imaginary part

$$\frac{1}{q_3} = \frac{L - \frac{\pi^2 w^4}{\lambda^2 f}\left(1 - \frac{L}{f}\right)}{L^2 + \frac{\pi^2 w^4}{\lambda^2}\left(1 - \frac{L}{f}\right)^2} - i\frac{\frac{\pi w^2}{\lambda}}{L^2 + \frac{\pi^2 w^4}{\lambda^2}\left(1 - \frac{L}{f}\right)^2}, \tag{20}$$

which we identify as

$$\frac{1}{q_3} = \frac{1}{R'} - i\frac{\lambda}{\pi w'^2}. \tag{21}$$

This lets us determine the distance of the focus from the lens. The wide-spread opinion that the focus has a distance equal to the focal lens is true only in some limit as we will see in the following. The focus is reached when the radius of curvature diverges, i.e. $1/R' = 0$, respectively when

$$L - \frac{\pi^2 w^4}{\lambda^2 f}\left(1 - \frac{L}{f}\right) = 0 \tag{22}$$

This leads leads to the distance of the focus

$$L = f\frac{\pi^2 \frac{w^4}{\lambda^4}}{\frac{f^2}{\lambda^2} + \pi^2 \frac{w^4}{\lambda^4}}. \tag{23}$$

Thus, the well-known formula $L = f$ is only valid if the incident beam waist is much larger than the wavelength, for $w \gg \lambda$. We calculate the beam waist w' in the focus by equating the imaginary parts of (20) and (21):

$$\frac{\lambda}{\pi w'^2} = \frac{\frac{\pi w^2}{\lambda}}{L^2 + \frac{\pi^2 w^4}{\lambda^2}\left(1 - \frac{L}{f}\right)^2}, \tag{24}$$

which can be solved for

$$w' = \sqrt{\frac{L^2 + \frac{\pi^2 w^4}{\lambda^2}\left(1 - \frac{L}{f}\right)^2}{\frac{\pi^2 w^2}{\lambda^2}}}. \tag{25}$$

In the approximation of $w \gg \lambda$, where the minimum is at $L = f$, the beam waist can be expressed by a simple formula:

$$w' = f \cdot \frac{\lambda}{\pi w}, \tag{26}$$

6 Exercise sheet 06, to be presented on 27.11.2025

6.1 Exc: Laser beams in cavities

(a) Starting from the beamwaist in a linear, asymmetric cavity (mirror radii R_1 and R_2, and cavity length L), calculate the curvature of the beam on one of the mirror surfaces. Use the equations from the lecture for the distance a of the beamwaist from the mirror, and the beamwaist w_0:

$$\begin{aligned} a &= \frac{L(R_2-L)}{R_1+R_2-2L}, \\ w_0 &= \sqrt{\frac{\lambda}{\pi}\cdot\sqrt{\frac{L(R_1-L)(R_2-L)(R_1+R_2-L)}{(R_1+R_2-2L)^2}}}. \end{aligned}$$

(b) Write a program to plot the beamwaist (formula above) as a function of the cavity length in the range $L \in [0, R_1+R_2]$. Consider the cases (i) with $R_1 = R_2 = 5\,\text{cm}$, and (ii) with $R_1 = 5\,\text{cm}$ and $R_2 = 6\,\text{cm}$, with wavelength $\lambda = 780\,\text{nm}$.

(c) Consider now a symmetric cavity with mirror radii of $R_1 = R_2 = 5\,\text{cm}$, and a distance of the mirrors of $L = 8\,\text{cm}$. Write a program to plot the beamwaist $w(z)$ in the cavity, but also outside the cavity after transmission through one of the cavity mirrors (ABCD matrices). The cavity mirrors consists of a glass substrate (refractive index $n = 1.5$) with thickness $d = 1\,\text{cm}$, where the front side (cavity-side) is curved with radius R and the back-side is flat.

Solution:

(a) The distance of beamwaist position from the first mirror is

$$a = \frac{L(R_2-L)}{R_1+R_2-2L},$$

and the beamwaist there is

$$w_0 = \sqrt{\frac{\lambda}{\pi}\cdot\frac{\sqrt{L(R_1-L)(R_2-L)(R_1+R_2-L)}}{R_1+R_2-2L}}.$$

At the beamwaist, the q-parameter is given by

$$q_0 = iz_0 = i\frac{\pi w_0^2}{\lambda} = i\frac{\sqrt{L(R_1-L)(R_2-L)(R_1+R_2-L)}}{R_1+R_2-2L}.$$

At the first mirror, the q-parameter is

$$q_a = iz_0 + a = i\frac{\sqrt{L(R_1-L)(R_2-L)(R_1+R_2-L)}}{R_1+R_2-2L} + \frac{L(R_2-L)}{R_1+R_2-2L}.$$

and

$$\begin{aligned} \frac{1}{q_a} &= \frac{1}{a+iz_0} = \frac{a-iz_0}{a^2+z_0^2} \\ &= \frac{\frac{L(R_2-L)}{R_1+R_2-2L} - i\frac{\sqrt{L(R_1-L)(R_2-L)(R_1+R_2-L)}}{R_1+R_2-2L}}{\left(\frac{L(R_2-L)}{R_1+R_2-2L}\right)^2 + \left(\frac{\sqrt{L(R_1-L)(R_2-L)(R_1+R_2-L)}}{R_1+R_2-2L}\right)^2} \\ &= \frac{1}{R_a} - i\frac{\lambda}{\pi w_a^2}. \end{aligned}$$

Thus,

$$\begin{aligned}
R_a &= \frac{\left(\frac{L(R_2-L)}{R_1+R_2-2L}\right)^2 + \left(\frac{\sqrt{L(R_1-L)(R_2-L)(R_1+R_2-L)}}{R_1+R_2-2L}\right)^2}{\frac{L(R_2-L)}{R_1+R_2-2L}} \\
&= \frac{L^2(R_2-L)^2 + L(R_1-L)(R_2-L)(R_1+R_2-L)}{L(R_2-L)(R_1+R_2-2L)} \\
&= \frac{L(R_2-L) + (R_1-L)(R_1+R_2-L)}{R_1+R_2-2L} \\
&= \frac{LR_2 - L^2 + R_1^2 - LR_1 + R_1R_2 - lR_2 - LR_1 + L^2}{R_1+R_2-2L} \\
&= \frac{R_1(R_1+R_2-2L)}{R_1+R_2-2L} \\
&= R_1
\end{aligned}$$

We see that, on the mirror surface, the curvature of the phase fronts equals the curvature of the mirror. This is actually true at both mirrors.

(b) The simulation leads to following results: We see that for the asymmetric cavity, there exists a

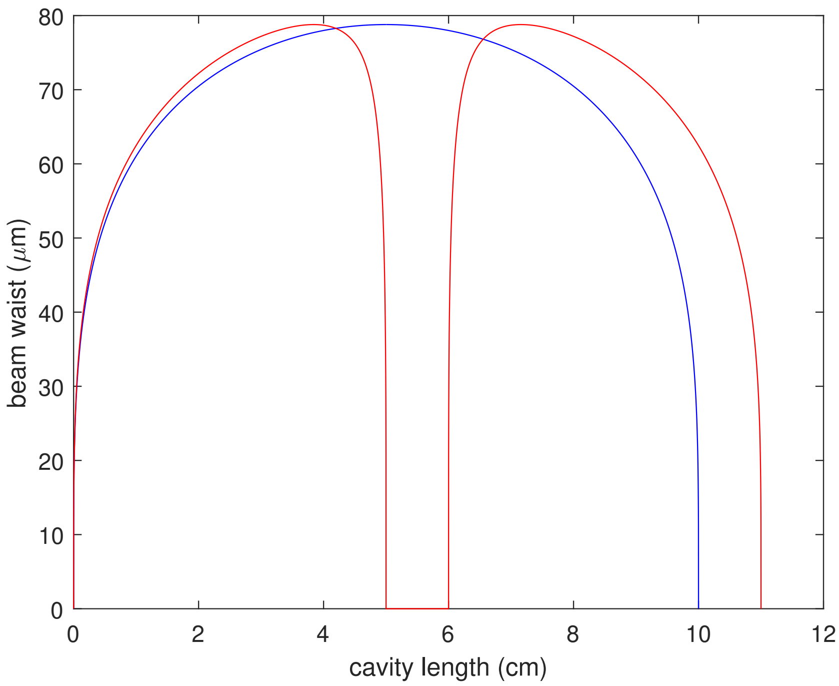

Figure 5: Beamwaist in a cavity for a (i) $R_1 = R_2 = 5\,\text{cm}$ (blue), and (ii) $R_1 = 5\,\text{cm}$, $R_2 = 6\,\text{cm}$ (red).

distance range from $L = 5\,\text{cm}$ to $L = 6\,\text{cm}$, where the cavity is unstable. Note, that one has to plot $\sqrt{\mathfrak{Re}\,(w_0^2)}$ to get the behavior in the unstable region correctly.

(c) We start at the beam waist in the cavity center with its complex q-parameter

$$q_0 = \frac{1}{\frac{1}{R_0} - i\pi\frac{\lambda}{w_0^2}} = i\frac{w_0^2}{\pi\lambda},$$

and take the beamwaist from the program in (b) as $w_0 = 70.467\,\mu\text{m}$. First, we calculate the propagation to the first mirror surface with ABCD matrix

$$M_1 = \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}, \rightarrow q(z) = \frac{q_0 + z}{1}$$

for $z = 0...4\,\text{cm}$. The last value $q(L/2)$ is transformed by the ABCD matrix for the curved interface

$$M_2 = \begin{pmatrix} 1 & 0 \\ \frac{n_1 - n_2}{R n_2} & \frac{n_1}{n_2} \end{pmatrix}, \rightarrow q_2(0) = \frac{q(L/2)}{-\frac{n_2 - n_1}{n_2 R_c} \cdot q(L/2) + \frac{n_2}{n_1}},$$

where the curvature of the mirror is taken as negative: $R_c = -5\,\text{cm}$, and $n_1 = 1.0$ and $n_2 = 1.5$. After that, free propagation in the glass substrate is calculated with the ABCD matrix form above. Then comes another interface, but now with $R_c = \infty$, which simplifies the calculation. Note the at the second interface $n_1 = 1.5$, and $n_2 = 1.0$. Finally, another free propagation follows. At each step. we calculate the beamwaist and radius of curvature via

$$\begin{aligned} w &= \sqrt{\frac{-\lambda}{\pi \cdot \mathfrak{Im}\,(1/q)}} \\ R &= \mathfrak{Re}\,(1/q). \end{aligned}$$

The simulations results are plotted in Fig. 6. We note that the inner surface of the mirror does not change the curvature. This comes from the fact that the cavity mode has the same radius of curvature as the mirror itself. However, the flat back-side of the mirror acts as a diverging lens which makes the outgoing beam diverge faster than if there was no such surface. The red-dashed line is calculated without the refraction at the mirror surfaces for comparison.

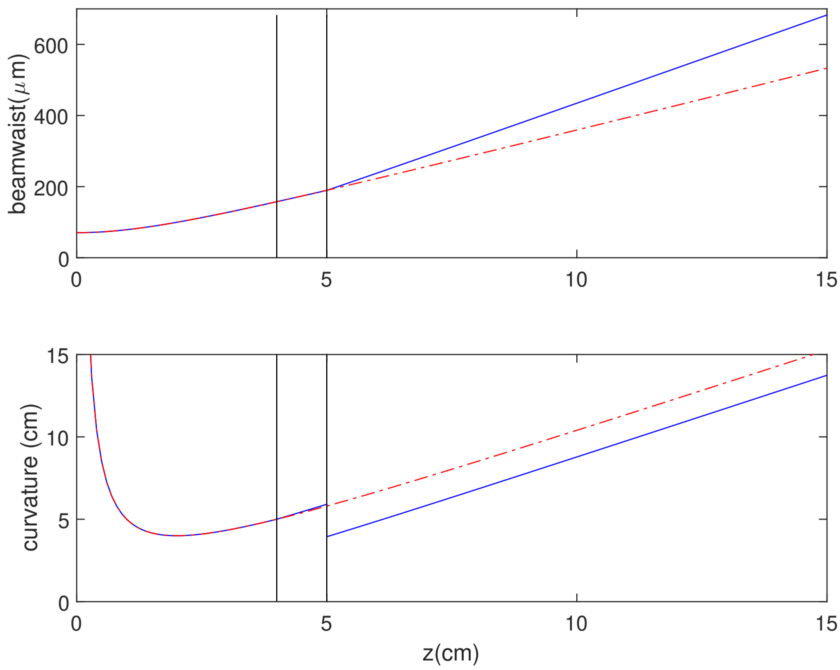

Figure 6: Beamwaist and radius of curvature of cavity beam and its transmission through the cavity mirror (blue line), and an hypothetic beam without transmission through the mirror surfaces (red dashed line).

6.2 Exc: Cavity power, spectrum and time dynamics

(a) Consider a linear cavity with mirror power reflectivities $R_1 = R_2 = R$, and no additional losses. Calculate first the transmission of light through both mirrors when the light would not show interference in the cavity. (This would be for instance the case, when the coherence length of the light was smaller than the cavity length.) Compare this value with the maximum and minimum transmission, as determined from the Airy function of the intra-cavity power.

(b) Consider a linear cavity with mirror power reflectivities $R_1 = R_2 = R = 0.995$. Calculate the finesse and the power enhancement in the cavity, for additional round-trip losses of $L_{\text{add}} = 0$, $L_{\text{add}} = 1-R$, and $L_{\text{add}} = 2 \cdot (1-R)$.

(c) Calculate the transverse oscillation frequency of a cavity with mirror radii of $R_{c1} = R_{c2} = 5\,\text{cm}$, and mirror distance $L = 8\,\text{cm}$. Make a sketch of the spectrum of a single free spectral range, with transverse modes up to $m + n = 3$. Label the modes with their quantum numbers.

(d) Ring-down measurement: A linear cavity with mirror distance $L = 8\,\text{cm}$ is pumped to reach the stationary state. A photodiode measures the transmitted power. When the pump light field is switched-off suddenly, the measured power decays exponentially. With a measured 1/e-power-decay time of $5\,\mu\text{s}$, what are the finesse $\mathcal{F}$ and the full-width-at-half-max δ_{fwhm} of the cavity?

Solution:

(a) Without interference, the transmission through both mirrors is given by $T = T_1 \cdot T_2 = (1 - R_1) \cdot (1 - R_2) = (1 - R)^2$. We compare this value with the max and min of the Airy function

$$\frac{P_c}{P_i} = \frac{t_1^2}{1 + r_m^2 - 2r_m \cos(\varphi)} = \frac{(1-R)}{1 + R^2 - 2R\cos(\varphi)},$$

where we have used that the round-trip amplitude reflectivity is $r_m = \sqrt{R_1}\sqrt{R_2} = R$. We have to take into account also for transmission through the second mirror to get

$$T(\varphi) = T\frac{P_c}{P_i} = \frac{(1-R)^2}{1 + R^2 - 2R\cos(\varphi)},$$

At the maximum ($\varphi = 0$), we get

$$T(0) = \frac{(1-R)^2}{1 + R^2 - 2R} = 1,$$

and at the minimum ($\varphi = \pi$), we get

$$T(\pi) = \frac{(1-R)^2}{1 + R^2 + 2R} = \frac{(1-R)^2}{(1+R)^2} \approx \frac{(1-R)^2}{4},$$

where the approximation is for $R \approx 1$.

(b) We use for the finesse

$$F = \frac{\pi}{\sqrt{r_m}(1 - r_m)}$$

and power enhancement

$$A = \frac{1 - R_1}{(1 - \sqrt{R_m})^2}$$

with $r_m = \sqrt{R_m} = \sqrt{R_1 R_2 (1 - L)} = \sqrt{R^2(1 - L)}$. The numerical results are for $R = 0.995$:

$$\begin{aligned} L = 0 \quad &\rightarrow \quad F = 640, A = 200 \\ L = 1 - R \quad &\rightarrow \quad F = 421, A = 89 \\ L = 2(1 - R) \quad &\rightarrow \quad F = 316, A = 50 \end{aligned}$$

(c) The transverse oscillation frequency is

$$\nu_t = \left[\frac{1}{\pi}\arccos(1-\frac{L}{R})\right]\cdot\nu_{\text{fsr}} \approx 0.7\cdot\nu_{\text{fsr}}$$

With resonance frequency

$$\nu_{pmn} = p\nu_{\text{fsr}} + (m+n+1)\cdot\nu_t$$

of modes with longitudinal quantum number p and transverse quantum number m, n, the spectrum looks like

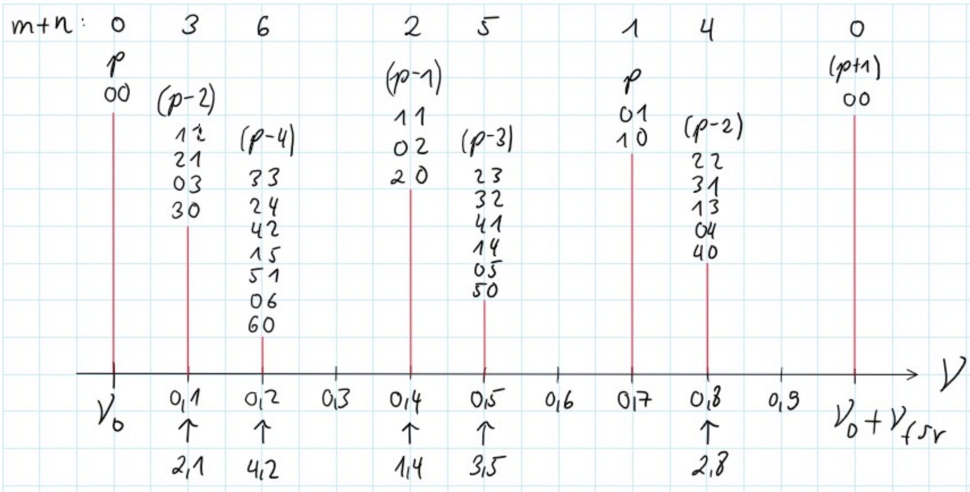

Figure 7: Cavity spectrum for a transverse oscillation frequency of $\nu_t = 0.7\cdot\nu_{\text{fsr}}$.

(d) We first calculate the full-width at half-maximum from the 1/e-power decay time τ as

$$\delta_{\text{fwhm}} = \frac{1}{2\pi\tau} = 31.8\,\text{kHz}.$$

The free-spectral range is

$$\nu_{\text{fsr}} = \frac{c}{2L} = 1.87\,\text{GHz},$$

from which we derive the finesse

$$F = \frac{\nu_{\text{fsr}}}{\delta_{\text{fwhm}}} = 58\,800.$$

6.3 Exc: Transfer matrix method

(a) Simulate numerically on the computer the reflectivity R of the dielectric mirror that has been introduced in the lecture as an example of the transfer matrix method. (Lecture 06, page 33). Plot $R(\lambda)$ for $\lambda \in [500\,\text{nm}, 800\,\text{nm}]$.

(b) Derive the Airy formula for the intracavity light power, using the transfer matrix formalism.

Solution:

(a) We use the amplitude Fresnel reflection r_{ij} and transmission t_{ij} coefficients at an interface from refractive index medium n_i to n_j with

$$r_{ij} = \frac{n_i - n_j}{n_i + n_j}, t_{ij} = \frac{2n_i}{n_i + n_j}$$

to derive the transfer matrix for the interface as

$$M_{ij} = \begin{pmatrix} t_{ij} - \frac{r_{ij}r_{ji}}{t_{ji}} & \frac{r_{ji}}{t_{ji}} \\ -\frac{r_{ij}}{t_{ji}} & \frac{1}{t_{ji}} \end{pmatrix} = \frac{1}{2n_j}\begin{pmatrix} n_j + n_i & n_j - n_i \\ n_j - n_i & n_j + n_i \end{pmatrix}.$$

The transfer matrix for the medium with refractive index n_i of length d is given by

$$M_i = \begin{pmatrix} \exp(i2\pi n_i d/\lambda) & 0 \\ 0 & \exp(-i2\pi n_i d/\lambda) \end{pmatrix}.$$

The overall matrix for the whole multilayer system is given by

$$M = M_{20} \cdot M_2 \cdot M_{12} \cdot M_1 \cdot (M_{21} \cdot M_2 \cdot M_{12} \cdot M_1)^4 \cdot M_{01},$$

from the components of which we get the reflection coefficient

$$r = \frac{M(1,2)}{M(2,2)}.$$

We plot $R = |r|^2$ as a function of the wavelength λ: The maximum reflection can be increased by

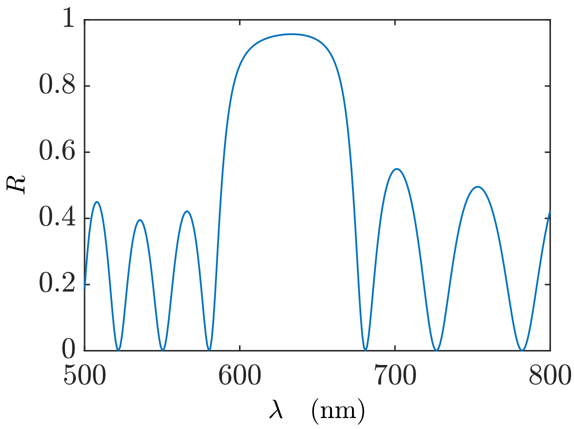

Figure 8: Power reflection R from a dielectric multilayer mirror.

adding more layers. To be more realistic, one could add also optical losses in the layers.

(b) We define the amplitude transmission and reflection coefficients of the two mirrors as $r_{12} = r$, $r_{21} = -r$, $r_{34} = -r$, $r_{43} = r$, resp. $t_{12} = t_{21} = t_{34} = t_{43} = t$. The transfer matrix at the incoupling mirror is given in the lecture:

$$\mathcal{T}_1 = \begin{pmatrix} t_{12} - \frac{r_{12}r_{21}}{t_{21}} & \frac{r_{21}}{t_{21}} \\ -\frac{r_{12}}{t_{21}} & \frac{1}{t_{21}} \end{pmatrix} = \frac{1}{t}\begin{pmatrix} 1 & -r \\ -r & 1 \end{pmatrix},$$

where we have used that $r^2 + t^2 = 1$. The propagation over the cavity length L is described by

$$\mathcal{T}_2 = \begin{pmatrix} \exp(i\varphi/2) & 0 \\ 0 & \exp(-i\varphi/2) \end{pmatrix},$$

where we have defined $\varphi = 2kL$ the same way as in the lecture. The second mirror is given by

$$\mathcal{T}_3 = \begin{pmatrix} t_{34} - \frac{r_{34}r_{43}}{t_{43}} & \frac{r_{43}}{t_{43}} \\ -\frac{r_{34}}{t_{43}} & \frac{1}{t_{43}} \end{pmatrix} = \frac{1}{t}\begin{pmatrix} 1 & r \\ r & 1 \end{pmatrix}.$$

The different sign in the non-diagonal terms, compared to the incoupling mirror, comes from the fact that we are now reflecting at the other side of the mirror. We calculate the matrix multiplication:

$$\mathcal{T} = \mathcal{T}_3 \cdot \mathcal{T}_2 \cdot \mathcal{T}_1 = \frac{1}{t^2}\begin{pmatrix} \exp(i\varphi/2) - r^2\exp(-i\varphi/2) & -r\exp(i\varphi/2) + r\exp(-i\varphi/2) \\ r\exp(i\varphi/2) - r\exp(-i\varphi/2) & -r^2\exp(i\varphi/2) + \exp(-i\varphi/2) \end{pmatrix}.$$

The overall transmission through the cavity is given by

$$T = \frac{1}{|\mathcal{T}_{22}|^2} = \frac{t^4}{(-r^2 + \exp(-i\varphi)) \cdot (-r^2 + \exp(i\varphi))} = \frac{t^4}{1 + r^4 - 2r^2\cos(\varphi)}.$$

This is equivalent to the result in the lecture, when we take into account that the round-trip reflectivity is given by $r_m = r^2$, and by considering the additional factor t^2 that relates the light power transmitted through the outcoupling mirror with the intracavity power.

7 Exercise sheet 07, to be presented on 04.12.2025

Solve the following exercises of the script: *Atom-Light Interactions*
Please download the latest version.

Exc.5.6.0.1 Quick ullage of an optical cavity

Exc.5.6.0.4 Cooperative amplification for a rubidium gas in a cavity

Exc.5.6.0.5 Characteristic parameters for various atom-cavity systems

Exc.5.6.0.6 Number of photons in a cavity

Exc.5.6.0.7 Derivation of the Heisenberg equations for the Jaynes-Cummings model